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REVIEW OF CIRCUIT SOLUTIONS LEADING TO DETERMINISTIC CHAOS IN OSCILLATORS BASED ON THE VAN DER POL EQUATION

The article is devoted to deterministic chaos generators based on the Van der Pol equation (VDP) and its modified versions. The papers that investigate the nonlinear behavior of the proposed modifications of the VDP equation, in particular, those that lead to chaotic behavior, are considered. Most purely mathematical works are not considered, preferring those that include modeling of the proposed systems of equations using simulators (PSpice, MultiSIM, OrCAD) and building a real prototype on an analog computer. The selected works are divided into three types of equations and generators based on them. The basic equation of VDP with an external signal source (forced) is studied at different signal parameters (shape, amplitude) and circuit modifications (adding passive elements, active filters, changing the number of external signal sources, nonlinear element). It is determined that robots based on the Van der Pol-Duffing equation are characterized by richer nonlinear behavior, even without an external signal source. The ability to control more system parameters leads to the observation of discontinuous oscillations, hyperchaos, and multistability. Both autonomous and non-autonomous versions of the schemes are presented. As the latter type, we study the Bonhoeffer-Van der Pol equation, which comes from the field of neuroscience, chemistry, and electrophysiology of the heart and can be converted into an electrical circuit. These implementations demonstrate interesting nonlinear behavior and deterministic chaos, but in a narrower range of system parameters. It is found that the system is characterized by fast-slow dynamics and has a complex mixed mode of oscillations. The result of the work is a non-exhaustive review of existing deterministic chaos generators based on the VDP equation, the existing gaps in the study of certain types of generators, approaches and methods for their creation, and tools used in their study are clarified. The structuring of existing approaches opens up the possibility of using the features of the above schemes to create completely new mathematical models or improve existing ones for covert information transmission systems.

Key words: *deterministic chaos, chaos generator, forced oscillator, strange attractor, multiscroll attractor, bursting patterns, jerk oscillator, hyperchaos, nonautonomous chaos, computer simulation, PSpice simulations, FPGA, ACRO, VDP, Van der Pol, TOVDP, MVDP, VDPD, MVDPD, BVP, NDR.*

Formulation of the problem. Some of the best known early examples of deterministic chaos from various fields are the Lorenz attractor (1963), Robert May's logic map (1976), Otto Rössler's system of differential equations (1976), and Chua's generator (1983). Such systems are well studied and are widely used both to consider nonlinear phenomena from a mathematical point of view and for applications such as in the field of secure communication [1] and to create various schemes for a chaotic signal generator [2] that can be used as a carrier. However, an earlier manifestation of deterministic chaos is known, discovered, but not realized at the time, in the work of Van der Pol (VDP) and Jan der Mark [3], and re-explored using the scheme only 60 years later [4]. Most of the available works investigating in detail the nonlinear processes in the VDP equation are purely mathematical, although a review of the available material shows that there are various circuit variants of the VDP relaxation oscillator that lead to rich nonlinear behavior, including chaos. In spite of this, it was not possible to find any review papers that provided in a convenient form, at least in an introductory format, a list of possible circuit solutions leading to chaotic behavior in VDP relaxation oscillators.

Analysis of recent research and publications. Most of the available publications focus on purely mathematical consideration of nonlinear behavior in different variants of the VDP oscillator. For example, the VDP equations are analyzed using the multiple scaling method (MSM) [5], a fractional model solution method [6], a mixed mode oscillator [7, 8], or an artificial intelligence based system for selecting optimal parameters for a forced VDP oscillator [9], as well as schemes using idealized elements [10]. There are fewer extensive works that examine the behavior of different circuits in simulation or on a real prototype, in [11] some chaos generators and synchronization methods are considered and practical applications are given. In [12], the use of analog computers (AC) is proposed for the visual study of chaotic systems. The connection between strange attractors and fractals, methods of fixing chaos are given in [13], and the only extensive and systematized collection of various schemes is the book [14], which, however, does not describe all possible variants of a chaos generator based on the VDP oscillator. The historical development of nonlinear oscillators, including relaxation oscillators, is devoted to [15].

Task statement. A brief and non-exhaustive survey of existing variants of circuit implementation of deterministic chaos generators based on the VDP relaxation generator equation and tested for the presence of chaotic behavior by means of simulation in MATLAB, SPICE software environments or prototyping. The classification was performed according to the modification of the VDP equation.

Outline of the main material of the study. In 1926, Balthasar van der Pol in [16] emphasizes a separate group of oscillations, which he calls "relaxation oscillations" (although they have been observed before in other works or in parallel to his [17]), and, most importantly, gives a dimensionless equation describing it (which allows us to use it not only in electrical engineering) and provides a circuit based on a triode. A number of robots appear in which the phenomenon of synchronization in a circuit with an external signal source is discussed [18, 19], and the paper [3] describes forced oscillations in a circuit with a neon lamp. Such a scheme was already known and led to relaxation oscillations, but in the scheme with an external source noise was observed before the transition to a new, lower frequency. Later Cartwright and Littlewood mathematically investigated the VDP equation and observed "strange behavior", which was a manifestation of chaotic dynamics [20, 21]. In 1961, using an analog computer, chaotic behavior was also observed in the VDP equation. It was observed by Ueda at some parameters of angular frequency [21]. Perhaps the first practical implementation of a scheme similar to the VDP oscillator, for which a phase porter of a chaotic attractor was built, was proposed by Pikovsky and Rabinovich [21]. After the "Chua's scheme" was created and the phenomenon of deterministic chaos was proved and shown [22, 23], an active development and search for schemes leading to chaos began. Thus, going back to [3], Kenedy and Chua [4] show the way to chaos through adding a period in a forced VDP oscillator, proving that the irregular noise at the transition to a new frequency was chaotic behavior. Thus there was a circuit proof that it is possible to design different versions of deterministic chaos generators using the VDP equation.

Briefly consider some works referred to different types depending on the equation, it should be taken into account that the authors of some of them combine different basic variants and one work can be referred to several types at once, also it should be specified that in the work under the word "chaos" and its derivatives there is exactly the phenomenon of deterministic chaos, not stochastic process, as well as under "analog computer" is meant the realization of equations with the help of digital amplifiers, analog multipliers, inverters.

Van der Pol's forced oscillators (VDP). With the development of semiconductors, one of the main ways to observe chaotic behavior using circuitry is to implement the equation using AC built on operational amplifiers (OA) and analog multipliers [12–14]. One such variant of forced VDP is given in [24]. This scheme allows controlling 4 parameters and produces many rich bifurcations leading to chaos. If the source of the sinusoidal signal is replaced by an oscillating circuit, you can get an autonomous version of the forced oscillator VDP (ACRO). However, it will not depend on the period of the tuned loop. A variant of such a circuit built on an op-amp is studied in [25].

In the VDP oscillator one can observe bursting oscillations [26] and, based on the fast-slow dynamics, observe chaotic behavior at some parameter values. The scheme implemented by AC for such a variant was presented in [27]. The forcing signal consists of two different parameterized slowly varying functions $\sin(x)$. One of them is fixed, in the other one the frequency changes at constant amplitude, and chaotic oscillations are observed at a frequency of 0.6–0.8 Hz.

The periodic excitation can also be replaced by a triangular or meander signal [28]. It is found that, at a large dissipation term ($\varepsilon = 5$), chaotic behavior occurs at lower frequencies compared to $\sin(x)$. The paper presents a circuit implemented on AC, numerical simulation, and experimental verification are presented. Quasi-periodic and chaotic attractors have been observed.

The realization of the equation is possible not only on AC, but also using a microcontroller. Thus, proposing a system of modified VDP oscillator, in which the potential of forced oscillation is proportional to the nonlinear term $\sin^n(x)$, and thus changing the power of the sinusoidal function, in [29] it was found that at certain values of the nonlinear dissipation term ε, it is possible to obtain, chaotic behavior even at its small values. The mathematical model was validated on the layout (Arduino Uno and oscilloscope). Developing their past work in [30], using meander and triangular signal and varying its power, the authors confirm chaotic, quasi-periodic and periodic behavior, also, besides mathematical model and validation on the layout, electronic simulation in OrCAD-Pspice is carried out. It is also possible to replace the source of sinusoidal signal by a pulse source [31, 32], the obtained strange attractors have a large number of turns and are called "Multiscrolls" [33], but the simulation of the electronic circuit in these works is not given, which opens the possibility of further investigation of this type of influence.

The problem of selecting a high-frequency nonlinear element arises in the design of microwave oscillators, one of the variants having the area of negative differential impedance is given in [34]. The authors consider three mathematical models used to describe oscillations in chaotic and periodic systems. Using a simplified VDP model they demonstrate the behavior of the proposed circuit on a resonant tunnel diode when multiplying and dividing the frequency of the input signal. The realization of the nonlinear element is possible based on a negative differential resistance (NDR) transistor structure. A variant of such a voltage controlled oscillator (VCO) based on the modified VDP equation is studied in [35]. Two cases of system dynamics are considered, in the presence and absence of white noise. The considered approach is useful in terms of real systems implementation and for oscillator synchronization. Developing this approach in [36], the authors propose new schemes of voltage-controlled VDP microwave oscillators on field-effect and bipolar transistors with NDR, and the possibility of tuning their frequency in different ways. More variants of VDP oscillators operating in relaxation and quasi-periodic modes are mathematically and experimentally investigated in [37].

An oscillator with a fifth order nonlinear term (restoring force potential φ^6) is studied in [38], a scheme of the proposed equation on AC is constructed in Multisim simulator, in which by changing the value of two resistors $R_{8\text{–}9}$ the numerical simulation of this equation is confirmed, as a result, the authors give phase portraits for double and triple spiral attractors obtained from two simulations and from the layout of the scheme.

Since Leon Chua proposed a new element, the memristor [39], various variants of its realization and use have gradually started to appear, due to its hysteresis V-I characteristic it is interesting to use it as a nonlinear element in schemes of deterministic chaos generation. One of such schemes of forced VDP is investigated in [40]. The circuits are implemented on the OA and analog multipliers, experimental verification at different frequencies of the periodic signal source has been carried out, the results converge with numerical simulations.

Another scheme using a memristor, but with two periodic signal sources (angular frequencies are in the ratio of 3 to 1), is investigated in [41]. The proposed memristor model is tested in Multisim, and the chaotic dynamics of the circuit is analyzed only numerically, based on resonance relations. The main focus of the paper is on the comparison of numerical integration methods, where symplectic methods are found to be more accurate and efficient.

By including an active low pass filter or all pass filter (APF) between the nonlinear element the authors of [42] present a variant of modified VPD (MVDP). The equation that describes it involves three differential equations so the authors define it as third order VDP oscillators (TOVDP), the scheme being autonomous due to this modification transitions to chaos without an external signal source. The paper studies antimonotonicity, coexisting attractors, hysteresis. Besides analytical and numerical verification, the authors make a prototype of the proposed scheme and confirm the existence of a strange attractor by changing two new parameters (responsible for the delay) introduced by the filter and a resistor in the feedback loop.

An interesting case, which shows the influence of model simplification on the dynamics of the system, is [43]. Considering the model proposed by Pikovsky and Rabinovich, the authors complicate it by bringing it to a more realistic form, adding an amplifying element on a transistor and a tunnel diode. Modeling is carried out in Pspice and Matlab-Simscape. Three variants of the model are considered, each with its own behavior (steady state, period doubling, chaos). However, for the last model, which is the closest to the real one, no attractor is given, only non-periodic oscillations. It is assumed that with a longer checking period, the observed state would have shifted to chaotic. The paper also considers synchronization through non-entropy by introducing a rectangular voltage source, which suppresses the chaotic behavior.

Oscillators Van der Pol–Duffing (VDPD). Another well-known nonlinear second-order differential equation leading to deterministic chaos in the presence of an external sinusoidal force is the Duffing oscillator, whose restoring force contains cubic nonlinearity. It is well studied from the mathematical point of view [44] and on its basis there are various circuit implementations [45, 46] investigating its nonlinear dynamics. Adding cubic nonlinearity to the equation of the forced VDP oscillator transforms it into the Van der Pol-Duffing equation, which combines auto-oscillatory behavior with strong nonlinearity, the dependence of the oscillation period on the amplitude (non-isochronous oscillation), more possibilities for control, rich chaotic behavior.

One variant of the scheme, which is a modified Shinriki scheme [47], was proposed in a series of papers [48, 49]. In the papers, the Takens-Bogdanov, Silnikov, and Hoph bifurcation, the coexistence of deterministic chaos and periodic modes, was found. The nonlinear element consists of an array of diodes and an operational amplifier to precisely control the unfolding parameter.

Another diode-based scheme is given in [50]. In addition to the idealized diode, the scheme including a cos(x) signal source and a negative resistance is a kind of forced oscillator VDPD. At certain parameters the limit cycle, torus and transition to a chaotic attractor are observed.

Let us consider several variants that slightly modify the basic scheme for these robots, the first one [51]. The changes consist in adding a resistor in parallel to one of the inductors, which allows us to consider dynamic processes in a small range of parameters. The circuit has one cubic nonlinear element. Numerical verification of the circuit using MATLAB and Fortran for chaotic behavior is given. The second [52] resistor is added in series with the inductor, in [53] combine series and parallel resistor and add a source of oscillation which makes the system non-autonomous, the dynamics is investigated at different parameters of the periodic signal. An interesting feature is the use of a two-parameter bifurcation spatial diagram obtained from Lyapunov spectra. It is concluded that a larger range of parameters is achieved leading to chaotic behavior in the absence of a parallel resistor in a non-autonomous circuit.

All the above robots study the VDPD circuit only from a mathematical point of view, without creating a prototype. In [54], a circuit without inductance parallel resistor is given, the nonlinear element is the same as in [48], modulation in PSpice is performed, and an experimental setup is created. The authors investigate the possibility of synchronizing two autonomous modified VDPD (MVDPD) by magnetic coupling and achieve zero synchronization errors.

A forced VPD oscillator with potential φ^8 (a nonlinear term of the seventh order appears in the equation of the restoring force potential) is proposed in [55]. Using the method of multiple scales, stability conditions for fixed points and resonant oscillations are given. The behavior of the system is examined using bifurcation diagrams, Lyapunov exponents, phase portraits, time series and Poincaré maps. To validate the numerical and analytical results, the system is verified in MultiSIM and in an experimental setup. Changing the parameter of resistor R_7 allows us to observe transitions between periodic and chaotic behavior.

A generator demonstrating discontinuous oscillations based on OA and analog multipliers to simulate a nonlinear element, when modeled in OrCAD-PSpice and numerically investigated, is given in [56]. The authors find symmetric bifurcations and coexisting attractors. A model for integer and fractional order form (positive Lyapunov exponents appear in a smaller range of parameters) of the studied oscillator is given, the peculiarity of which is the choice of the range of control coefficients, the work focuses on the case for double hump potential (for $\alpha > 0$ and $\beta < 0$). A mathematical study of a jerk VPD with external and parametric forcing is given in [57], but the authors do not verify their model by means of a circuit.

By replacing the periodic source in a non-autonomous VDPD by positive feedback, one can reduce the generator size [58] and obtain chaotic behavior (the case of double hump potential) in a certain range of parameters. The verification of the proposed system of equations is performed in Orcard-PSpice. The circuit is assembled on an op-amp and analog multipliers. The cubic nonlinearity is changed by changing the parameter of resistor R_{14}.

A nonlinear system with symmetry breaking by adding an appropriate parameter to the VDPD equation (affects the bias current of the OA) and exhibiting multistability, five coexisting attractors (including hidden attractors) is proposed in [59]. The numerical results are verified using circuit modulation (analog computer) in PSpice.

Using the fourth order differential equation in [60], a system with hyperchaos property is given, there are two positive Lyapunov exponents, belonging to MVPD. To verify the numerical results, it is implemented on field programmable gate array

(FPGA), software-hardware cosimulation is used and direct Euler method is used as the mathematical model, Kintex-7 processor is selected for the hardware.

Oscillators Bonhoeffer–van der Pol (BVP). Another modification of the VPD equation is the Bonhoeffer-van der Pol equation, which is a simplified FitzHugh-Nagumo model [61] and is used to describe excitable systems, particularly in neuroscience and cardiac electrophysiology. It is less complex (in terms of the number of control parameters) than VDPD, although it has cubic nonlinearity. Its feature is fast-slow dynamics and mixed mode oscillation (MMO) [62, 63, 64]. At certain values of the terms of the equation also exhibits chaotic behavior (in forced systems). There are few works that describe the schemes and experimentally verify them, mainly mathematical models are given.

An analog circuit of a forced oscillator BVP with a nonlinear element built on an op-amp and diodes is experimentally investigated in [65]. The authors observe a complex mixed mode of oscillations with incrementing bifurcations (MMOIB) and discontinuous chaotic MMO (which is often observed in chemical experiments) in a narrow range of parameters. The dynamics of the coupled system is mathematically investigated in [66], where the case of series and ring coupling of two and three forced BVPs, the possibility of synchronization of the systems and the transition from chaos to limit cycle are tested. Experimentally the scheme with resistively coupled BVP oscillators is proposed in [67] under a weak periodic perturbation. By changing the coupling strength in a small range of parameters a strange attractor is observed.

Conclusions. Some number of variants of deterministic chaos realization in schemes based on the VPD equation has been considered. It is supposed that by combining the considered solutions it is possible to offer new variants of schemes. The examination is not exhaustive, but only shows the variety of approaches that can be used to observe complex nonlinear dynamics. By modifying the basic relaxation oscillation equation proposed by VPD, adding a nonlinear element and an external signal source, one can obtain a forced oscillator; a restoring force containing a cubic nonlinearity – VDPD; a slow recovery variable and a cubic nonlinearity – BVP. In the above works, a large number of ways to influence the dynamics of the system have been proposed: the shape or power of the signal source; the number of signal sources; the restoring force (potential φ); the type of nonlinear element; changing the ratings of circuit elements (or the values of the terms of the equation in the case of an analog computer); the number of oscillators; the coupling coefficient; the detuning coefficient; active filters; additional passive elements. Thus one can observe different dynamics: periodic; quasi-periodic; chaotic behavior; different variants of bifurcations; MMOIB; intermittency; torus collapse; multiscrolls; discontinuous oscillations; hyperchaos; multistability; crises; symmetric and asymmetric attractors.

The study of the dynamics of systems in the works is carried out numerically, analytically, by modulating the circuit in a simulator, experimentally on a prototype. In most cases, the circuits are built using an analog computer (op-amp, analog multipliers, inverters) or on a microcontroller, FPGA. Less often on passive elements and those with a nonlinear V-I characteristic. The authors of most works do not create a real prototype, limiting themselves to simulation, which, as shown in some articles, can distort the real results and give only approximate values at which there is a change in the dynamics of the system.

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Семенов А. О., Хльоба А. А. ОГЛЯД СХЕМОТЕХНІЧНИХ РІШЕНЬ, ЩО ПРИЗВОДЯТЬ ДО ДЕТЕРМІНОВАНОГО ХАОСУ В ГЕНЕРАТОРАХ НА ОСНОВІ РІВНЯННЯ ВАН ДЕР ПОЛЯ

Стаття присвячена генераторам детермінованого хаосу побудованих на основі рівняння Ван дер Поля (VDP) і його модифікованих версіях. Розглянуто роботи, в яких досліджується нелінійна поведінка запропонованих модифікацій рівняння VDP, зокрема така що призводить до хаотичної поведінки. Більшість винятково математичних робіт не розглядається, віддаючи перевагу тим, що включають моделювання запропонованих систем рівнянь за допомогою симуляторів (PSpice, MultiSIM, OrCAD) і побудову реального прототипу на аналоговому комп'ютері. Виокремлені праці розділені за трьома типами рівнянь і генераторами на їх основі. Базове рівняння VDP із зовнішнім джерелом сигналу (вимушене) досліджується при різних параметрах сигналу (форма, амплітуда) та модифікаціях схеми (додавання пасивних елементів, активних фільтрів, зміна кількісті джерел зовнішнього сигналу, нелінійного елемента). Визначено, що роботи на основі рівняння Van der Pol-Duffing відрізняються багатшою нелінійною поведінкою, навіть без зовнішнього джерела сигналу. Можливість управління більшою кількістю параметрів системи призводить до спостереження розривних коливань, гіперхаосу, мультистабільністі. Наведено роботи як автономних, так і неавтономних варіантів схем. Як останній тип досліджується рівняння Bonhoeffer-Van der Pol, що походить із галузі нейронаук, хімії та електрофізіології серця, та може бути перетворене на електричну схему. Наведені реалізації *демонструють цікаву нелінійну поведінку і детермінований хаос, але у вужчому діапазоні параметрів системи. З'ясовано, що система відрізняється динамікою «швидко-повільно», має складний змішаний режим коливань. Результатом виконаної роботи є невичерпний огляд існуючих генераторів детермінованого хаосу на основі рівняння VDP, з'ясовані наявні прогалини в дослідженні певних типів генераторів, підходи і методи їх створення, інструменти які використовуються при їх досліджені. Структурування існуючих підходів відкриває можливість використання особливостей наведених схем для створення абсолютно нових математичних моделей або поліпшення наявних для систем прихованої передачі інформації.*

Ключові слова: *детермінований хаос, генератор хаосу, вимушений гененратор, дивний аттрактор, розривні коливання, багатовитковий атрактор, гіперхаос, неавтономний хаос, комп'ютерне моделювання, моделювання PSpice, ПЛІС, ACRO, Van der Pol, VDP, TOVDP, MVDP, VDPD, MVDPD, BVP, NDR.*